

On certain integral functionals of squared Bessel processes

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Abstract

Let X be a squared Bessel process. Following a Feynman-Kac approach, the Laplace transforms of joint laws of $(U, \int_0^{R_y} X_s^p ds)$ are studied where R_y is the first hitting time of y by X and U is a random variable measurable with respect to the history of X until R_y . A subset of these results are then used to solve the associated small ball problems for $\int_0^{R_y} X_s^p ds$ and determine a Chung's law of iterated logarithm. $(\int_0^{R_y} X_s^p ds)$ is also considered as a purely discontinuous increasing Markov process and its infinitesimal generator is found. The findings are then used to price a class of exotic derivatives on interest rates and determine the asymptotics for the prices of some put options that are only slightly in-the-money.

Key words: Bessel processes, modified Bessel functions, first passage times, small deviations, Chung's law of iterated logarithm, non-homogeneous Feller jump process, time reversal, last passage times, subordinator, interest rate derivatives.

1 Introduction

Let X be a squared Bessel process which is the unique strong solution to

$$dX_t = 2(\nu + 1) dt + 2\sqrt{X_t} dB_t,$$

where $\nu \geq -1$ is a real constant and B is a standard Brownian motion. Letting $\delta = 2(\nu + 1)$, X is called a δ -dimensional squared Bessel process. We will denote such a process with $X_0 = x$ by $BESQ^\delta(x)$ and δ and ν will be related by $\delta = 2(\nu + 1)$ throughout the text. In this paper we are interested in the integral functional

$$\Sigma_{p,x,y}^\delta := \int_0^{R_y} X_s^p ds, \quad (1.1)$$

where $p > -1$ and $R_y := \inf\{t \geq 0 : X_t = y\}$ for $y \in [0, \infty)$ (In the sequel, we will write R_y^δ only if we need to specify the dimension to avoid ambiguity.) and X is $BESQ^\delta(x)$.

Squared Bessel processes have found wide applications especially in Finance Theory, see Chapter 6 in [7] for a recent account. They can, e.g., be used to model interest rates in a *Cox-Ingersoll-Ross* framework. In the above setting, if X^p models the spot interest rates, then $\exp(\Sigma_{p,x,y}^\delta)$ refers to the cumulative interest until the spot rate hits the barrier y^p . As such, this random variable is related to certain *exotic options* on interest rates (see [4] for some formulae regarding barrier options in a similar framework). Bessel processes also appear often in the study of *financial bubbles* since $1/\sqrt{X}$ is the prime example of a continuous (strict) local martingale when X is a $BESQ(3)$ (see, e.g., [13], [15] and [16] for how strict local martingales, and in particular Bessel processes, appear in mathematical studies of bubbles).

In Section 2 we will determine the joint law of $(U, \Sigma_{p,x,y}^\delta)$ by martingale methods, where U is a random variable measurable with respect to the evolution of X until R_y . In particular we will obtain the joint distributions of $(R_y, \Sigma_{p,x,y}^\delta)$ and $(\max_{t \leq R_y} X_t, \Sigma_{p,x,y}^\delta)$. As a by-product of our findings, if $\frac{|\nu|}{p+1} = \frac{1}{2}$, we have a remarkable characterisation of the conditional law of $\Sigma_{p,x,y}^\delta$ when $x \geq y$ (resp. $x \leq y$) given that the maximum (resp. minimum) of X at R_y is below (resp. above) a fixed level in terms of the first hitting distributions of a 3-dimensional Bessel process.

We will use the results of Section 2 in order to study *small ball probabilities* for $\Sigma_{p,x,y}^\delta$ in Section 3. Solving the small ball problem for $\Sigma_{p,x,y}^\delta$ amounts to finding the asymptotic behaviour of $-\log \text{Prob}(\Sigma_{p,x,y}^\delta < \varepsilon)$ as $\varepsilon \rightarrow 0$. We will then use this asymptotic form to determine a *law of iterated logarithm* for $(\Sigma_{p,0,y}^\delta)_{y \geq 0}$ as $y \rightarrow \infty$.

Section 4 will analyse $(\Sigma_{p,0,y}^\delta)_{y \geq 0}$ as a Markov process indexed by y and compute its infinitesimal generator when $\nu \geq 0$. We will also consider the process Z^δ which is obtained via a ‘time reversal’ from $(\Sigma_{p,0,y}^\delta)_{y \geq 0}$. More precisely, we will find the generator of Z^δ defined by

$$Z_x^\delta = \int_{L_1-x}^{L_1} X_s^p ds \quad \forall x \in [0, 1),$$

where $L_x := \sup\{t \geq 0 : X_t = x\}$. In particular, we will obtain that Z^4 is identical in law to an increasing family of hitting times of a linear Brownian motion.

Finally, in Section 5 we will apply our findings to the pricing of some exotic derivatives on interest rates. The small ball probabilities will be used to find asymptotic behaviour of some put options with small strikes, the options that are only slightly *in-the-money*.

2 Preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a stochastic base where \mathcal{F} is completed with the \mathbb{P} -null sets. Let X be an \mathbb{R}_+ -valued semimartingale which is the unique strong solution to

$$dX_t = 2(\nu + 1) dt + 2\sqrt{X_t} dB_t, \quad (2.1)$$

where $\nu \geq -1$ is a real constant, B is a standard Brownian motion.

$\mathcal{X} = (\Omega, \mathcal{F}, (\mathcal{F}_t), (X_t), (Q_x^\delta)_{x \in \mathbb{R}_+})$ is a Markov process with values in \mathbb{R}_+ . Here Q_x^δ is the law of X starting at x . It is well-known (see Chapter IX of [14]) that for $\nu \geq 0$ the set $\{0\}$ is polar, otherwise it is reached a.s.. Moreover, the process is transient for $\nu > 0$ and recurrent otherwise. We will denote the first hitting time of 0 for X with R . The scale function, s^ν , for $BESQ^\delta$ is given by

$$s^\nu(x) = -x^{-\nu} \quad \text{for } \nu > 0, \quad s^0(x) = \log x, \quad s^\nu(x) = x^{-\nu} \quad \text{for } \nu \in [-1, 0).$$

We refer the reader to [14] and [5] for a comprehensive study of Bessel processes and relevant bibliography.

In subsequent computations we will follow a Feynman-Kac type approach as in, e.g., [6].

Lemma 2.1 *Let $p > -1, \lambda > 0$ and suppose that $u \in C^2$, solves the following ordinary differential equation (ODE):*

$$x^2 y'' + xy' - y[\nu^2 + \lambda x^{2(p+1)}] = 0, \quad (2.2)$$

and is strictly positive on $(0, \infty)$. Then, $(M_{t \wedge R}^{(u)})_{t \geq 0}$ is a local martingale where

$$M_t^{(u)} := u(\sqrt{X_t}) X_t^{-\frac{\nu}{2}} \exp \left(-\frac{\lambda}{2} \int_0^t X_s^p ds \right).$$

In particular,

$$\mathbf{1}_{[t < R]} dM_t^{(u)} = \mathbf{1}_{[t < R]} M_t^{(u)} \left(\frac{u'(\sqrt{X_t})}{u(\sqrt{X_t})} - \frac{\nu}{\sqrt{X_t}} \right) dB_t.$$

PROOF. If we let $w(x) := u(\sqrt{x}) x^{-\frac{\nu}{2}}$, it is easily seen that w solves

$$2xw'' + 2(\nu + 1)w' - \frac{\lambda}{2}x^p w = 0. \quad (2.3)$$

Thus, on $[t < R]$

$$dM_t^{(u)} = 2M_t^{(u)} \frac{w'}{w}(X_t) \sqrt{X_t} dB_t = M_t^{(u)} \left(\frac{u'(\sqrt{X_t})}{u(\sqrt{X_t})} - \frac{\nu}{\sqrt{X_t}} \right) dB_t.$$

■

The solutions to (2.2) can easily be determined via the modified Bessel functions, I_α and

K_α , of the first and second kind. Recall that (see, e.g., Section 3.7 in [17]) I_α (resp. K_α) is an increasing (resp. decreasing) solution to

$$x^2 y'' + xy' - (x^2 + \alpha^2)y = 0. \quad (2.4)$$

Moreover, they have the following series representation:

$$I_\alpha(x) = \sum_{m=0}^{\infty} \frac{1}{m! \Gamma(m + \alpha + 1)} \left(\frac{x}{2}\right)^{2m+\alpha} \quad K_\alpha(x) = \frac{\pi}{2} \frac{I_{-\alpha}(x) - I_\alpha(x)}{\sin(\alpha\pi)}. \quad (2.5)$$

There exist also integral representations (see p. 172 of [17]) as follows:

$$I_\alpha(x) = \frac{\left(\frac{1}{2}x\right)^\alpha}{\Gamma\left(\alpha + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_{-1}^1 (1-t^2)^{\alpha-\frac{1}{2}} e^{-xt} dt, \quad \alpha > -\frac{1}{2}; \quad (2.6)$$

$$K_\alpha(x) = \frac{\Gamma\left(\frac{1}{2}\right) \left(\frac{1}{2}x\right)^\alpha}{\Gamma\left(\alpha + \frac{1}{2}\right)} \int_1^\infty e^{-xt} (t^2 - 1)^{\alpha-\frac{1}{2}} dt, \quad \alpha > -\frac{1}{2}. \quad (2.7)$$

Then, it is easy to check that the solutions to (2.2) is of the form

$$C_1 K_{\frac{|\nu|}{p+1}} \left(\frac{1}{p+1} \sqrt{\lambda} x^{p+1} \right) + C_2 I_{\frac{|\nu|}{p+1}} \left(\frac{1}{p+1} \sqrt{\lambda} x^{p+1} \right), \quad (2.8)$$

where C_1 and C_2 are arbitrary constants.

We now return to determining the joint law of $(U, \Sigma_{p,x,y}^\delta)$ for arbitrary positive \mathcal{F}_{R_y} -measurable random variables U , where $\Sigma_{p,x,y}^\delta$ is as defined in (1.1). We will analyse the cases of negative and positive ν separately.

2.1 The case $\nu < 0$

Theorem 2.1 *Suppose that $\nu < 0$, $p > -1$ and let $u_0(x) := K_{\frac{-\nu}{p+1}} \left(\frac{1}{p+1} \sqrt{\lambda} x^{p+1} \right)$. Then, $(M_{t \wedge R}^{(u_0)})_{t \geq 0}$ is a bounded martingale with*

$$\mathbf{1}_{[t < R]} dM_t^{(u_0)} = \mathbf{1}_{[t < R]} M_t^{(u_0)} \left(\frac{u'(\sqrt{X_t})}{u(\sqrt{X_t})} - \frac{\nu}{\sqrt{X_t}} \right) dB_t.$$

PROOF. In order to show the boundedness property it suffices to show that

$$\lim_{x \rightarrow 0} u_0(\sqrt{x}) x^{-\frac{\nu}{2}} < \infty, \quad \lim_{x \rightarrow \infty} u_0(\sqrt{x}) x^{-\frac{\nu}{2}} < \infty.$$

To see that the first limit is finite, apply a change of variable $u = xt$ in (2.7) to get for $\alpha > 0$

$$x^\alpha K_\alpha(x) = \frac{\Gamma\left(\frac{1}{2}\right) \frac{1}{2^\alpha}}{\Gamma\left(\alpha + \frac{1}{2}\right)} \int_x^\infty e^{-u} (u^2 - x^2)^{\alpha-\frac{1}{2}} du, \quad (2.9)$$

which converges to a finite limit as $x \rightarrow 0$. This implies that $\lim_{x \rightarrow 0} u_0(\sqrt{x})x^{-\frac{\nu}{2}} < \infty$.

In order to find the limit at infinity, we will make another use of the integral representation in (2.7). Note that, by substituting $t = 1 + \frac{u}{x}$, (2.7) turns into

$$K_\alpha(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} \frac{e^{-x}}{\Gamma\left(\alpha + \frac{1}{2}\right)} \int_0^\infty e^{-u} u^{\alpha-\frac{1}{2}} \left(1 + \frac{u}{2x}\right)^{\alpha-\frac{1}{2}} du, \quad (2.10)$$

due to which we conclude that

$$\lim_{x \rightarrow \infty} u_0(\sqrt{x})x^{-\frac{\nu}{2}} < \infty.$$

In view of Lemma 2.1 we now obtain that $(M_{t \wedge R}^{(u_0)})_{t \geq 0}$ is a bounded martingale. ■
It is well-known that (see, e.g. Section 2.8 in [12]) for $\nu > -1$

$$Q_x^\delta \left[\exp \left(-\frac{\lambda}{2} R_y \right) \right] = \frac{x^{-\frac{\nu}{2}} K_\nu(\sqrt{\lambda x})}{y^{-\frac{\nu}{2}} K_\nu(\sqrt{\lambda y})}, \quad y \leq x; \quad (2.11)$$

$$= \frac{x^{-\frac{\nu}{2}} I_\nu(\sqrt{\lambda x})}{y^{-\frac{\nu}{2}} I_\nu(\sqrt{\lambda y})}, \quad y \geq x. \quad (2.12)$$

Note that the above formulas are still valid when $\nu \geq 0$.

Since $R < \infty$, a.s. when $\nu < 0$, the following is a straightforward corollary to the theorem above for $\nu < 0$ and $y \leq x$.

Corollary 2.1 *Let u_0 be the function defined in Theorem 2.1 and suppose that $\nu < 0$, $p > -1$ and $y \leq x$. If U is \mathcal{F}_{R_y} -measurable, then for $r \geq 0$,*

$$Q_x^\delta \left[\exp \left(-rU - \frac{\lambda}{2} \Sigma_{p,x,y}^\delta \right) \right] = \frac{u_0(\sqrt{x})}{u_0(\sqrt{y})} \left(\frac{x}{y} \right)^{-\frac{\nu}{2}} P_x^{\delta, u_0} [\exp(-rU)],$$

where P_x^{δ, u_0} is defined by $\frac{P_x^{\delta, u_0}}{Q_x^\delta} = M_R^{(u_0)}$. Moreover, under P_x^{δ, u_0} , X solves

$$dX_t = 2 \left(\frac{u'_0(\sqrt{X_t})\sqrt{X_t}}{u_0(\sqrt{X_t})} + 1 \right) dt + 2\sqrt{X_t} d\beta_t, \quad t \leq R, \quad (2.13)$$

for some P_x^{δ, u_0} -Brownian motion β .

Remark 1 *Note that $u_0(x)x^{-\nu}$ is decreasing. This follows from the recurrence relation (see Section 3.7 in [17])*

$$K'_\alpha(x) = -\frac{\alpha}{x} K_\alpha(x) - K_{\alpha-1}(x).$$

Remark 2 *By taking $U \equiv 0$ Corollary 2.1 yields the law of $\Sigma_{p,x,y}^\delta$ for $y \leq x$. Comparing this Laplace transform with (2.11) shows that $\Sigma_{p,x,y}^\delta \stackrel{d}{=} R_{y^*}^{\delta^*}$ where $\delta^* = 2(\frac{\nu}{1+p} + 1)$, $y^* = \frac{y^{1+p}}{(1+p)^2}$ and $R_{y^*}^{\delta^*}$ is the first hitting time of y^* for some BESQ $^{\delta^*}(\frac{x^{1+p}}{(1+p)^2})$. Moreover, this equality in law would be valid for $y \geq x$, too. These facts also follow from the time-change result given in Proposition XI.1.1 in [14].*

In particular if $U = R_y$, we obtain the following in view of the well-known formulas for the Laplace transform of the hitting times of one-dimensional diffusions:

Corollary 2.2 *Let u_0 be the function defined in Theorem 2.1 and suppose that $\nu < 0$ and $y \leq x$. Then for $r \geq 0$,*

$$Q_x^\delta \left[\exp \left(-rR_y - \frac{\lambda}{2} \Sigma_{p,x,y}^\delta \right) \right] = \frac{u_0(\sqrt{x})}{u_0(\sqrt{y})} \left(\frac{x}{y} \right)^{-\frac{\nu}{2}} \frac{\Phi(x)}{\Phi(y)},$$

where Φ is a continuous and decreasing solution of

$$2xv'' + 2 \left(\frac{u'_0(\sqrt{x})\sqrt{x}}{u_0(\sqrt{x})} + 1 \right) v' = rv. \quad (2.14)$$

Corollary 2.1 also allows us to compute the law of $\Sigma_{p,x,y}^\delta$ on the event that a certain boundary is yet to be reached.

Corollary 2.3 *Let u_0 be the function defined in Theorem 2.1 and suppose that $\nu < 0$, $p > -1$ and $y \leq x$. Then, a scale function of the diffusion defined by (2.13) is*

$$\tilde{s}_0(x) = \int_1^x \frac{1}{yu_0^2(\sqrt{y})} dy, \quad x \geq 0. \quad (2.15)$$

Thus, for any $a > x$

$$Q_x^\delta \left[\mathbf{1}_{[R_a > R_y]} \exp \left(-\frac{\lambda}{2} \Sigma_{p,x,y}^\delta \right) \right] = \frac{u_0(\sqrt{x})}{u_0(\sqrt{y})} \left(\frac{x}{y} \right)^{-\frac{\nu}{2}} \frac{\tilde{s}_0(x) - \tilde{s}_0(a)}{\tilde{s}_0(y) - \tilde{s}_0(a)}.$$

PROOF. The representation of the scale function is due to the well-known formulas for the solutions of SDEs, see, e.g., Exercise VII.3.20 in [14]. Note that the function is well-defined at $x = 0$. Indeed, it follows from (2.9) that $\lim_{y \rightarrow 0} u_0(\sqrt{y})y^{-\frac{\nu}{2}} > 0$. Thus, $\lim_{y \rightarrow 0} yu_0^2(\sqrt{y}) > Cy^{1+\nu}$ for some $C > 0$. Since $y^{-(1+\nu)}$ is integrable for $\nu \in [-1, 0)$, the claim holds.

The second assertion follows from Corollary 2.1 after taking $r = 1$ and $U = \log \mathbf{1}_{[R_a > R_y]}$ via the defining property of scale functions, see Definition VII.3.3 in [14]. \blacksquare

Remark 3 *The above result in fact gives us the joint law of $(\max_{t \leq R_y} X_t, \Sigma_{p,x,y}^\delta)$. Indeed, for any $a \geq x$*

$$[\max_{t \leq R_y} X_t < a] = [R_a > R_y].$$

Since $K_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}}e^{-x}$, we have more explicit formulas when $\frac{\nu}{p+1} = -\frac{1}{2}$.

Corollary 2.4 *Suppose that $\frac{\nu}{p+1} = -\frac{1}{2}$ and $p > -1$. Then, for $y \leq x$ we have the following:*

i)

$$Q_x^\delta \left[\exp \left(-rR_y - \frac{\lambda}{2} \Sigma_{p,x,y}^\delta \right) \right] = \exp \left(\sqrt{\lambda} \frac{x^{-\nu} - y^{-\nu}}{2\nu} \right) \frac{\Phi(x)}{\Phi(y)},$$

where Φ is a continuous and decreasing solution of

$$2xv'' + 2 \left(\nu + 1 + \sqrt{\lambda} x^{-\nu} \right) v' = rv. \quad (2.16)$$

ii) The function \tilde{s}_0 is, up to an affine transformation, given by $\exp \left(-\sqrt{\lambda} \frac{x^{-\nu}}{\nu} \right)$.

iii) For $a > x$,

$$\begin{aligned} Q_x^\delta \left[\mathbf{1}_{[R_a > R_y]} \exp \left(-\frac{\lambda}{2} \Sigma_{p,x,y}^\delta \right) \right] &= \exp \left(\sqrt{\lambda} \frac{x^{-\nu} - y^{-\nu}}{2\nu} \right) \frac{\exp \left(-\sqrt{\lambda} \frac{x^{-\nu}}{\nu} \right) - \exp \left(-\sqrt{\lambda} \frac{a^{-\nu}}{\nu} \right)}{\exp \left(-\sqrt{\lambda} \frac{y^{-\nu}}{\nu} \right) - \exp \left(-\sqrt{\lambda} \frac{a^{-\nu}}{\nu} \right)} \\ &= \frac{\sinh \left(-\sqrt{\lambda} \frac{a^{-\nu} - x^{-\nu}}{2\nu} \right)}{\sinh \left(-\sqrt{\lambda} \frac{a^{-\nu} - y^{-\nu}}{2\nu} \right)}. \end{aligned} \quad (2.17)$$

Note that the expression in (2.17) yields

$$Q_x^\delta \left[\exp \left(-\frac{\lambda}{2} \Sigma_{p,x,y}^\delta \right) \middle| R_a > R_y \right] = \frac{a^{-\nu} - y^{-\nu}}{a^{-\nu} - x^{-\nu}} \frac{\sinh \left(-\sqrt{\lambda} \frac{a^{-\nu} - x^{-\nu}}{2\nu} \right)}{\sinh \left(-\sqrt{\lambda} \frac{a^{-\nu} - y^{-\nu}}{2\nu} \right)}$$

using the scale function of X under Q_x^δ . Comparing this with (2.12) for $\nu = 1/2$ gives the following

Corollary 2.5 Suppose that $\frac{\nu}{p+1} = -\frac{1}{2}$ and $p > -1$. Then, for $y \leq x < a$ we have that the law of $\Sigma_{p,x,y}^\delta$ conditioned on the event $[R_a > R_y]$ is that of the first hitting time of $(a^{-\nu} - y^{-\nu})^2/4\nu^2$ by a 3-dimensional squared Bessel process started at $(a^{-\nu} - x^{-\nu})^2/4\nu^2$.

Note that, since $\lim_{a \rightarrow 0} \frac{1-x^a}{a} = -\log x$, when $y > 0$, we obtain that the above conditional laws converge as $\nu \rightarrow 0$ (and, thus, as $p \rightarrow -1$) to that of the first hitting time of $(\log \sqrt{a} - \log \sqrt{y})^2$ by a 3-dimensional squared Bessel process started at $(\log \sqrt{a} - \log \sqrt{x})^2$. This can be viewed as the analogous statement of the above corollary when $\nu = 0$ and $p = -1$.

Next we look at the case when $y \geq x \geq 0$. Observe that the function u_1 as defined in the theorem below is still well defined and finite at $x = 0$ in view of, e.g., the series representation of I_α in (2.5).

Theorem 2.2 Suppose either that $p \geq 0$ and $-1 < \nu < 0$ or that $\nu = -1$ and $p > 0$. Let $u_1(x) := I_{\frac{\nu}{p+1}} \left(\frac{\sqrt{\lambda}}{p+1} x^{p+1} \right)$. Then, $(M_{t \wedge R_y}^{(u_1)})_{t \geq 0}$ is a bounded martingale with

$$\mathbf{1}_{[t < R_y]} dM_t^{(u_1)} = \mathbf{1}_{[t < R_y]} M_t^{(u_1)} \left(\frac{u_1'(\sqrt{X_t})}{u_1(\sqrt{X_t})} - \frac{\nu}{\sqrt{X_t}} \right) dB_t,$$

for any $y \geq 0$.

PROOF. Note that $[R < R_y]$ has a positive probability. Thus, we have to pay attention to the behaviour of $w(x) = u_1(\sqrt{x})x^{-\frac{\nu}{2}}$ at $x = 0$. Observe that under our assumptions, $\frac{\nu}{p+1} > -1$, thus it follows from the series representation of I_α that $w(0) > 0$ and is finite since $I_\alpha(1) < \infty$ for any α . Next, we will show that w has an absolutely continuous derivative over $[0, \infty)$. Using the recurrence relations (see Section 3.7 in [17])

$$\begin{aligned} I_{\alpha-1}(x) + I_{\alpha+1}(x) &= 2I'_\alpha(x), \text{ and} \\ \frac{x}{2}(I_{\alpha-1}(x) - I_{\alpha+1}(x)) &= \alpha I_\alpha(x), \end{aligned}$$

we obtain that

$$I'_\alpha(x) = I_{\alpha+1}(x) + \frac{\alpha}{x}I_\alpha(x). \quad (2.18)$$

Using this identity it follows from direct calculations that

$$w'(x) = \frac{\sqrt{\lambda}}{2}x^{\frac{p-\nu-1}{2}}I_\gamma\left(\frac{\sqrt{\lambda}}{p+1}x^{\frac{p+1}{2}}\right),$$

where $\gamma = 1 + \frac{\nu}{p+1}$. Since the leading term of $I_\gamma\left(\frac{\sqrt{\lambda}}{p+1}x^{\frac{p+1}{2}}\right)$ as $x \rightarrow 0$ is $x^{\frac{\nu+p+1}{2}}$, we see that $\lim_{x \rightarrow 0} w'(x) = 0$ when $p > 0$. Therefore, we obtain immediately from the ODE (2.3) that when $p > 0$ $\lim_{x \rightarrow 0} xw''(x) = 0$ for $\nu \in [-1, 0)$. On the other hand, when $p = 0$ and $\nu > -1$,

$$\frac{w'(x)}{w(x)} = \frac{\lambda}{2} \frac{1}{\sqrt{\lambda x}} \frac{I_{\nu+1}(\sqrt{\lambda x})}{I_\nu(\sqrt{\lambda x})}$$

by another application of (2.18). Thus, in view of the series representation of I_α

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{w'(x)}{w(x)} &= \frac{\lambda}{2} \lim_{x \rightarrow 0} \frac{I_{\nu+1}(x)}{x I_\nu(x)} \\ &= \frac{\lambda \Gamma(\nu+1)}{4 \Gamma(\nu+2)} = \frac{\lambda}{4(\nu+1)}. \end{aligned}$$

Consequently, $\lim_{x \rightarrow \infty} 2(\nu+1)w' - \frac{\lambda}{2}w = 0$ since $w(x) > 0$ for all $x \geq 0$. Again, it follows from the ODE (2.3) that $\lim_{x \rightarrow 0} xw''(x) = 0$. However, this condition implies that $\int_0^x w''(y) dy$ exists and is finite. Since this integral equals $w'(x) - w'(0)$ for any $x \in [0, \infty)$, we conclude that w' is absolutely continuous on $[0, \infty)$ and $w'(x) = w'(0) + \int_0^x w''(y) dy$ for any $x \in [0, \infty)$. Then, in view of Problem 3.7.3 in [8] we immediately deduce that

$$\begin{aligned} w(X_t) &= w(X_0) + \int_0^t 2w'(X_s)\sqrt{X_s}dB_s + \int_0^t \{2w'(X_s)(\nu+1) + 2w''(X_s)X_s\} ds \\ &= \int_0^t 2w'(X_s)\sqrt{X_s}dB_s + \frac{\lambda}{2} \int_0^t X_s^p w(X_s) ds. \end{aligned}$$

A simple application of integration by parts formula now shows that $M^{(u_1)}$ is a martingale with the claimed representation. ■

Corollary 2.6 *Let u_1 be the function defined in Theorem 2.2 and suppose that the hypotheses therein hold. Then, we have the following for all $x \leq y$:*

i) *If $r \geq 0$ and U is positive and \mathcal{F}_{R_y} -measurable,*

$$Q_x^\delta \left[\exp \left(-rU - \frac{\lambda}{2} \Sigma_{p,x,y}^\delta \right) \right] = \frac{u_1(\sqrt{x})}{u_1(\sqrt{y})} \left(\frac{x}{y} \right)^{-\frac{\nu}{2}} P_x^{\delta,u_1} [\exp(-rU)],$$

where P_x^{δ,u_1} is defined by $\frac{P_x^{\delta,u_1}}{Q_x^\delta} = M_{R_y}^{(u_1)}$. Moreover, under P_x^{δ,u_1} , X solves

$$dX_t = 2 \left(\frac{u_1'(\sqrt{X_t})\sqrt{X_t}}{u_1(\sqrt{X_t})} + 1 \right) dt + 2\sqrt{X_t} d\beta_t, \quad t \leq R_y, \quad (2.19)$$

for some P_x^{δ,u_1} -Brownian motion β .

ii) *For all $r \geq 0$*

$$Q_x^\delta \left[\exp \left(-rR_y - \frac{\lambda}{2} \Sigma_{p,x,y}^\delta \right) \right] = \frac{u_1(\sqrt{x})}{u_1(\sqrt{y})} \left(\frac{x}{y} \right)^{-\frac{\nu}{2}} \frac{\Psi(x)}{\Psi(y)},$$

where Ψ is a continuous and increasing solution of

$$2xv'' + 2 \left(\frac{u_1'(\sqrt{x})\sqrt{x}}{u_1(\sqrt{x})} + 1 \right) v' = rv. \quad (2.20)$$

iii) *A scale function of the diffusion defined in (2.19) is given by*

$$\tilde{s}_1(x) = \int_1^x \frac{1}{yu_1^2(\sqrt{y})} dy, \quad x \geq 0. \quad (2.21)$$

Thus, for any $0 \leq a < x$

$$Q_x^\delta \left[\mathbf{1}_{[R_a > R_y]} \exp \left(-\frac{\lambda}{2} \Sigma_{p,x,y}^\delta \right) \right] = \frac{u_1(\sqrt{x})}{u_1(\sqrt{y})} \left(\frac{x}{y} \right)^{-\frac{\nu}{2}} \frac{\tilde{s}_1(x) - \tilde{s}_1(a)}{\tilde{s}_1(y) - \tilde{s}_1(a)}.$$

PROOF. The proof follows the same lines as in the proofs of analogous results for $y \leq x$, hence, is omitted. ■

Again, since $I_{-\frac{1}{2}}(x) = \sqrt{\frac{2\pi}{x}} \cosh(x)$ we have

Corollary 2.7 *$p \geq 0$ and $\frac{\nu}{p+1} = -\frac{1}{2}$. Then, for $y \geq x$ we have the following:*

i)

$$Q_x^\delta \left[\exp \left(-rR_y - \frac{\lambda}{2} \Sigma_{p,x,y}^\delta \right) \right] = \frac{\cosh \left(-\frac{\sqrt{\lambda}}{2\nu} x^{-\nu} \right) \Psi(x)}{\cosh \left(-\frac{\sqrt{\lambda}}{2\nu} y^{-\nu} \right) \Psi(y)},$$

where Ψ is a continuous and increasing solution of

$$2xv'' + 2 \left(\nu + 1 + \sqrt{\lambda} x^{-\nu} \tanh \left(-\frac{\sqrt{\lambda}}{2\nu} x^{-\nu} \right) \right) v' = rv. \quad (2.22)$$

ii) The function \tilde{s}_1 is, up to an affine transformation, given by $\tanh \left(-\sqrt{\lambda} \frac{x^{-\nu}}{2\nu} \right)$.

iii) For $0 \leq a < x$,

$$Q_x^\delta \left[\mathbf{1}_{[R_a > R_y]} \exp \left(-\frac{\lambda}{2} \Sigma_{p,x,y}^\delta \right) \right] = \frac{\sinh \left(-\sqrt{\lambda} \frac{a^{-\nu} - x^{-\nu}}{2\nu} \right)}{\sinh \left(-\sqrt{\lambda} \frac{a^{-\nu} - y^{-\nu}}{2\nu} \right)}. \quad (2.23)$$

PROOF. Only part iii) needs proof. Note that

$$Q_x^\delta \left[\mathbf{1}_{[R_a > R_y]} \exp \left(-\frac{\lambda}{2} \Sigma_{p,x,y}^\delta \right) \right] = \frac{\cosh \left(-\frac{\sqrt{\lambda}}{2\nu} x^{-\nu} \right) \tanh \left(-\frac{\sqrt{\lambda}}{2\nu} x^{-\nu} \right) - \tanh \left(-\frac{\sqrt{\lambda}}{2\nu} a^{-\nu} \right)}{\cosh \left(-\frac{\sqrt{\lambda}}{2\nu} y^{-\nu} \right) \cosh \left(-\frac{\sqrt{\lambda}}{2\nu} y^{-\nu} \right) - \tanh \left(-\frac{\sqrt{\lambda}}{2\nu} a^{-\nu} \right)}.$$

On the other hand,

$$\begin{aligned} & \cosh \left(-\frac{\sqrt{\lambda}}{2\nu} x^{-\nu} \right) \left\{ \tanh \left(-\frac{\sqrt{\lambda}}{2\nu} x^{-\nu} \right) - \tanh \left(-\frac{\sqrt{\lambda}}{2\nu} a^{-\nu} \right) \right\} \\ &= \frac{\sinh \left(-\frac{\sqrt{\lambda}}{2\nu} x^{-\nu} \right) \cosh \left(-\frac{\sqrt{\lambda}}{2\nu} a^{-\nu} \right) - \cosh \left(-\frac{\sqrt{\lambda}}{2\nu} x^{-\nu} \right) \sinh \left(-\frac{\sqrt{\lambda}}{2\nu} a^{-\nu} \right)}{\cosh \left(-\frac{\sqrt{\lambda}}{2\nu} a^{-\nu} \right)} \\ &= \frac{\sinh \left(-\frac{\sqrt{\lambda}}{2\nu} x^{-\nu} \right) \cosh \left(\frac{\sqrt{\lambda}}{2\nu} a^{-\nu} \right) + \cosh \left(-\frac{\sqrt{\lambda}}{2\nu} x^{-\nu} \right) \sinh \left(\frac{\sqrt{\lambda}}{2\nu} a^{-\nu} \right)}{\cosh \left(-\frac{\sqrt{\lambda}}{2\nu} a^{-\nu} \right)} \\ &= \frac{\sinh \left(\frac{\sqrt{\lambda}}{2\nu} (a^{-\nu} - x^{-\nu}) \right)}{\cosh \left(-\frac{\sqrt{\lambda}}{2\nu} a^{-\nu} \right)}, \end{aligned}$$

which yields the claimed representation. ■

Note that in fact we do not need to assume $p > 0$ for part iii) of the above result to hold since X is never 0 before R_a for $0 < a < x$ and also that (2.23) and (2.17) are the same. Thus,

Corollary 2.8 *Suppose that $\frac{\nu}{p+1} = -\frac{1}{2}$ and $p > -1$. Then, for $y \geq x > a$ we have that the law of $\Sigma_{p,x,y}^\delta$ conditioned on the event $[R_a > R_y]$ is that of the first hitting time of $(a^{-\nu} - y^{-\nu})^2/4\nu^2$ by a 3-dimensional squared Bessel process started at $(a^{-\nu} - x^{-\nu})^2/4\nu^2$.*

We end this section with a scaling property which will be useful in the subsequent section. It is a direct consequence of the scaling property of $BESQ^\delta$ applied to the definition of $\Sigma_{p,0,y}^\delta$.

Proposition 2.1 *Suppose that $p \geq 0, \nu < 0$. Then, we have the following identity in law for any $y \geq 0$:*

$$y^{p+1}\Sigma_{p,0,1}^\delta \stackrel{d}{=} \Sigma_{p,0,y}^\delta.$$

2.2 The case $\nu \geq 0$

Recall that when $\nu \geq 0$ the point 0 is polar for X . Thus, one can prove without any difficulty that $M^{(u_1)}$, where u_1 is as defined in Theorem 2.2, is a martingale stopped at R_y .

Theorem 2.3 *Suppose that $p > -1, \nu \geq 0$. Let u_1 be the function defined in Theorem 2.2. Then, $(M_{t \wedge R_y}^{(u_1)})_{t \geq 0}$ is a bounded martingale.*

Recall that $BESQ^\delta$ is transient when $\nu > 0$, thus $Q^x(R_y < \infty) = 1$ whenever $y \geq x$. Consequently, we can deduce the following.

Corollary 2.9 *Let u_1 be the function defined in Theorem 2.2 and suppose that $p > -1, \nu \geq 0$. Then, we have the following for all $x \leq y$:*

i) *If $r \geq 0$ and U is positive and \mathcal{F}_{R_y} -measurable,*

$$Q_x^\delta \left[\exp \left(-rU - \frac{\lambda}{2} \Sigma_{p,x,y}^\delta \right) \right] = \frac{u_1(\sqrt{x})}{u_1(\sqrt{y})} \left(\frac{x}{y} \right)^{-\frac{\nu}{2}} P_x^{\delta, u_1} [\exp(-rU)],$$

where P_x^{δ, u_1} is defined by $\frac{P_x^{\delta, u_1}}{Q_x^\delta} = M_{R_y}^{(u_1)}$. Moreover, under P_x^{δ, u_1} , X satisfies (2.19).

ii) *For all $r \geq 0$*

$$Q_x^\delta \left[\exp \left(-rR_y - \frac{\lambda}{2} \Sigma_{p,x,y}^\delta \right) \right] = \frac{u_1(\sqrt{x})}{u_1(\sqrt{y})} \left(\frac{x}{y} \right)^{-\frac{\nu}{2}} \frac{\Psi(x)}{\Psi(y)},$$

where Ψ is a continuous and increasing solution of (2.20).

iii) *For any $a < x$*

$$Q_x^\delta \left[\mathbf{1}_{[R_a > R_y]} \exp \left(-\frac{\lambda}{2} \Sigma_{p,x,y}^\delta \right) \right] = \frac{u_1(\sqrt{x})}{u_1(\sqrt{y})} \left(\frac{x}{y} \right)^{-\frac{\nu}{2}} \frac{\tilde{s}_1(x) - \tilde{s}_1(a)}{\tilde{s}_1(y) - \tilde{s}_1(a)},$$

where \tilde{s}_1 is as defined in (2.21).

Analogous to Proposition 2.1 we have the following scaling property.

Proposition 2.2 *Suppose that $p > -1, \nu \geq 0$. Then, we have the following identity in law for any $y \geq 0$:*

$$y^{p+1} \Sigma_{p,0,1}^\delta \stackrel{d}{=} \Sigma_{p,0,y}^\delta.$$

We now return to the case $y \leq x$.

Proposition 2.3 *Let u_0 be the function defined in Theorem 2.1 and suppose that $\nu \geq 0, p > -1$. Then, we have the following for all $x \geq y$:*

i) *If $r \geq 0$ and U is positive and \mathcal{F}_{R_y} -measurable,*

$$Q_x^\delta \left[\exp \left(-rU - \frac{\lambda}{2} \Sigma_{p,x,y}^\delta \right) \right] = \frac{u_0(\sqrt{x})}{u_0(\sqrt{y})} \left(\frac{x}{y} \right)^{-\frac{\nu}{2}} P_x^{\delta, u_0} [\exp(-rU)],$$

where P_x^{δ, u_0} is defined by $\frac{P_x^{\delta, u_0}}{Q_x^\delta} = M_{R_y}^{(u_0)}$. Moreover, under P_x^{δ, u_0} , X satisfies (2.13) until R_y .

ii) *For all $r \geq 0$*

$$Q_x^\delta \left[\exp \left(-rR_y - \frac{\lambda}{2} \Sigma_{p,x,y}^\delta \right) \right] = \frac{u_0(\sqrt{x})}{u_0(\sqrt{y})} \left(\frac{x}{y} \right)^{-\frac{\nu}{2}} \frac{\Phi(x)}{\Phi(y)},$$

where Φ is a continuous and decreasing solution of (2.14).

iii) *For any $a > x$*

$$Q_x^\delta \left[\mathbf{1}_{[R_a > R_y]} \exp \left(-\frac{\lambda}{2} \Sigma_{p,x,y}^\delta \right) \right] = \frac{u_0(\sqrt{x})}{u_0(\sqrt{y})} \left(\frac{x}{y} \right)^{-\frac{\nu}{2}} \frac{\tilde{s}_0(x) - \tilde{s}_0(a)}{\tilde{s}_0(y) - \tilde{s}_0(a)},$$

where \tilde{s}_0 is as defined in (2.15).

PROOF. We will only prove i). Note that $(M_{t \wedge R}^{(u_0)})_{t \geq 0}$ is no longer a uniformly integrable martingale since u_0 now explodes at $x = 0$. Nevertheless, $(M_{t \wedge R_y}^{(u_0)})_{t \geq 0}$ is still a uniformly integrable martingale. Thus, using the optional stopping theorem, we obtain

$$\begin{aligned} u_0(\sqrt{x}) x^{-\frac{\nu}{2}} P_x^{\delta, u_0} [\exp(-rU)] &= Q_x^\delta \left[\exp(-rU) M_{R_y}^{(u_0)} \right] \\ &= Q_x^\delta \left[\mathbf{1}_{[R_y < \infty]} u_0(\sqrt{y}) y^{-\frac{\nu}{2}} \exp \left(-rU - \frac{\lambda}{2} \Sigma_{p,x,y}^\delta \right) \right] \\ &\quad + Q_x^\delta \left[\mathbf{1}_{[R_y = \infty]} M_\infty^{(u_0)} \right]. \end{aligned}$$

For $\nu = 0$, R_y is finite a.s., hence the claim. In case of $\nu > 0$, we still obtain the formula since $u_0(\infty) = 0$, and $\Sigma_{p,x,y}^\delta = \infty$ on the set $[R_y = \infty]$ for X being transient. \blacksquare

As before, using the explicit form of $K_{\frac{1}{2}}$, one gets

Corollary 2.10 Suppose that $\frac{\nu}{p+1} = \frac{1}{2}$ and $p > -1$. Then, for $y \leq x$ we have the following:

i)

$$Q_x^\delta \left[\exp \left(-rR_y - \frac{\lambda}{2} \Sigma_{p,x,y}^\delta \right) \right] = \frac{y^\nu}{x^\nu} \exp \left(-\sqrt{\lambda} \frac{x^\nu - y^\nu}{2\nu} \right) \frac{\Phi(x)}{\Phi(y)},$$

where Φ is a continuous and decreasing solution of

$$2xv'' + 2 \left(-\nu + 1 + \sqrt{\lambda}x^\nu \right) v' = rv. \quad (2.24)$$

ii) The function \tilde{s}_0 is, up to an affine transformation, given by $\exp \left(\sqrt{\lambda} \frac{x^\nu}{\nu} \right)$.

iii) For $a > x$

$$\begin{aligned} Q_x^\delta \left[\mathbf{1}_{[R_a > R_y]} \exp \left(-\frac{\lambda}{2} \Sigma_{p,x,y}^\delta \right) \right] &= \frac{y^\nu}{x^\nu} \exp \left(-\sqrt{\lambda} \frac{x^\nu - y^\nu}{2\nu} \right) \frac{\exp \left(\sqrt{\lambda} \frac{x^\nu}{\nu} \right) - \exp \left(\sqrt{\lambda} \frac{a^\nu}{\nu} \right)}{\exp \left(\sqrt{\lambda} \frac{y^\nu}{\nu} \right) - \exp \left(\sqrt{\lambda} \frac{a^\nu}{\nu} \right)} \\ &= \frac{y^\nu \sinh \left(\sqrt{\lambda} \frac{a^\nu - x^\nu}{2\nu} \right)}{x^\nu \sinh \left(\sqrt{\lambda} \frac{a^\nu - y^\nu}{2\nu} \right)}. \end{aligned} \quad (2.25)$$

As in the case with $\frac{\nu}{p+1} = -\frac{1}{2}$ we get

$$\begin{aligned} Q_x^\delta \left[\exp \left(-\frac{\lambda}{2} \Sigma_{p,x,y}^\delta \right) \middle| R_a > R_y \right] &= \frac{y^\nu y^{-\nu} - a^{-\nu} \sinh \left(\sqrt{\lambda} \frac{a^\nu - x^\nu}{2\nu} \right)}{x^\nu x^{-\nu} - a^{-\nu} \sinh \left(\sqrt{\lambda} \frac{a^\nu - y^\nu}{2\nu} \right)} \\ &= \frac{1 - \left(\frac{y}{a} \right)^\nu \sinh \left(\sqrt{\lambda} \frac{a^\nu - x^\nu}{2\nu} \right)}{1 - \left(\frac{x}{a} \right)^\nu \sinh \left(\sqrt{\lambda} \frac{a^\nu - y^\nu}{2\nu} \right)} \\ &= \frac{a^\nu - y^\nu \sinh \left(\sqrt{\lambda} \frac{a^\nu - x^\nu}{2\nu} \right)}{a^\nu - x^\nu \sinh \left(\sqrt{\lambda} \frac{a^\nu - y^\nu}{2\nu} \right)}, \end{aligned}$$

and hence

Corollary 2.11 Suppose that $\frac{\nu}{p+1} = \frac{1}{2}$ and $p > -1$. Then, for $y \leq x < a$ we have that the law of $\Sigma_{p,x,y}^\delta$ conditioned on the event $[R_a > R_y]$ is that of the first hitting time of $(a^\nu - y^\nu)^2/4\nu^2$ by a 3-dimensional squared Bessel process started at $(a^\nu - x^\nu)^2/4\nu^2$.

Similarly, since $I_{\frac{1}{2}}(x) = \sqrt{\frac{2\pi}{x}} \sinh(x)$ we have

Corollary 2.12 $p > -1$ and $\frac{\nu}{p+1} = \frac{1}{2}$. Then, for $y \geq x$ we have the following:

i)

$$Q_x^\delta \left[\exp \left(-r R_y - \frac{\lambda}{2} \Sigma_{p,x,y}^\delta \right) \right] = \frac{y^\nu \sinh \left(\frac{\sqrt{\lambda}}{2\nu} x^\nu \right) \Psi(x)}{x^\nu \sinh \left(\frac{\sqrt{\lambda}}{2\nu} y^\nu \right) \Psi(y)},$$

where Ψ is a continuous and increasing solution of

$$2xv'' + 2 \left(-\nu + 1 + \sqrt{\lambda} x^\nu \coth \left(\frac{\sqrt{\lambda}}{2\nu} x^\nu \right) \right) v' = rv. \quad (2.26)$$

ii) The function \tilde{s}_1 is, up to an affine transformation, given by $\coth \left(\sqrt{\lambda} \frac{x^\nu}{2\nu} \right)$.

iii) For $0 \leq a < x$,

$$Q_x^\delta \left[\mathbf{1}_{[R_a > R_y]} \exp \left(-\frac{\lambda}{2} \Sigma_{p,x,y}^\delta \right) \right] = \frac{\sinh \left(\sqrt{\lambda} \frac{a^\nu - x^\nu}{2\nu} \right)}{\sinh \left(\sqrt{\lambda} \frac{a^\nu - y^\nu}{2\nu} \right)}. \quad (2.27)$$

Remark 4 Comparing parts i) of Corollary 2.4 and 2.10 immediately gives us that, for $x \geq y$, the distributions of $\Sigma_{p,x,y}^\delta$ are different when ν has different signs. On the other hand, Corollaries 2.5 and 2.11 imply that they have the same distribution once they are conditioned on the event that the maximum of the underlying squared Bessel process is less than a by time R_y . Same conclusion holds when $x \leq y$.

3 Small ball problem and Chung's law of iterated logarithm

The small ball problem (also called small deviations) for a stochastic process $Z = (Z_t)_{t \in \mathcal{T}}$ consists in finding the probability

$$\mathbb{P}[\|Z\| < \varepsilon] \quad \text{as } \varepsilon \rightarrow 0,$$

$\|\cdot\|$ is a given norm, usually L^p or L^∞ . It is connected to many other questions, such as the law of the iterated logarithm of Chung type (Chung's LIL for short), strong limit laws in statistics, metric entropy properties of linear operators and several approximation quantities for stochastic processes. The determination of the above probability is not feasible other than in a very few cases and one is inclined to consider the asymptotic behaviour of

$$-\log \mathbb{P}[\|Z\| < \varepsilon] \quad \text{as } \varepsilon \rightarrow 0.$$

The solution to the latter problem is also not available in full generality. However, one can get this asymptotic behaviour for Gaussian processes (see, e.g., [11] and [10]) or real-valued

Lévy processes (see [1]). There is a large amount of literature on small ball probabilities in the Gaussian setting and one can consult the survey article [11].

As one can expect from the computations made in the previous section, we will be interested in the small ball probabilities for the stochastic process $(X_t)_{t \geq 0}$, and the “norm”

$$\|Z\|_{p,y} = \left(\int_0^{R_y} |Z_t|^p dt \right)^{\frac{1}{p}},$$

where $p \in (0, \infty)$ and $R_y = \inf\{t > 0 : X_t = y\}$. Observe that the above definition is not a real norm unless $p \geq 1$, however, as the results in this section does not depend on whether $\|\cdot\|_{p,y}$ is a true norm, this is not a problem. Our results and proofs are close in nature to the results of [9].

Interestingly, the small ball probabilities for X under the above norm does not depend on its index, ν , as seen from the next theorem.

Theorem 3.1 *Let X be a BESQ $^\delta$ as defined by (2.1) and $R_y = \inf\{t > 0 : X_t = y\}$. Then, one has, for $x \geq 0$ and $y \geq 0$,*

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} \lambda^{-\frac{1}{2}} \log Q_x^\delta [\exp(-\lambda \|X\|_{p,y}^p)] &= -\frac{\sqrt{2}}{p+1} \left| x^{\frac{p+1}{2}} - y^{\frac{p+1}{2}} \right| \\ \lim_{\varepsilon \rightarrow 0} \varepsilon^p \log Q_x^\delta [\|X\|_{p,y} < \varepsilon] &= -\frac{1}{2(p+1)^2} \left(x^{\frac{p+1}{2}} - y^{\frac{p+1}{2}} \right)^2. \end{aligned}$$

PROOF. Let $w(x) = u(\sqrt{x})x^{-\frac{\nu}{2}}$ where $u = u_0$ (resp. $u = u_1$) for $y \leq x$ (resp. $y > x$) and u_0 and u_1 are as defined in Theorems 2.1 and 2.2, respectively. Then, it follows from the results of the previous section that

$$\sqrt{2}\lambda^{-\frac{1}{2}} \log Q_x^\delta \left[\exp \left(-\frac{\lambda}{2} \|X\|_{p,y}^p \right) \right] = \sqrt{2} \frac{\log w(x)}{\sqrt{\lambda}} - \sqrt{2} \frac{\log w(y)}{\sqrt{\lambda}}.$$

Moreover, when $u = u_0$,

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} \frac{\log w(x)}{\sqrt{\lambda}} &= \lim_{\lambda \rightarrow \infty} \frac{\log u_0(\sqrt{x})}{\sqrt{\lambda}} \\ &= \frac{x^{\frac{p+1}{2}}}{p+1} \lim_{\lambda \rightarrow \infty} \frac{\log K_{\frac{|\nu|}{p+1}} \left(\frac{\sqrt{\lambda}}{p+1} x^{\frac{p+1}{2}} \right)}{\sqrt{\lambda}} \frac{p+1}{x^{\frac{p+1}{2}}} \\ &= \frac{x^{\frac{p+1}{2}}}{p+1} \lim_{\lambda \rightarrow \infty} \frac{\log K_{\frac{|\nu|}{p+1}}(\lambda)}{\lambda}. \end{aligned}$$

However, using the integral representation (2.10) in the proof of Theorem 2.1, we obtain that for any $\alpha > -\frac{1}{2}$,

$$\log K_\alpha(x) = C - x - \frac{1}{2} \log x + \log \int_0^\infty e^{-u} u^{\alpha-\frac{1}{2}} \left(1 + \frac{u}{2x} \right) du.$$

Thus, $\lim_{\lambda \rightarrow \infty} \frac{\log K_{\frac{|\nu|}{p+1}}(\lambda)}{\lambda} = -1$ since

$$\lim_{x \rightarrow \infty} \int_0^\infty e^{-u} u^{\frac{|\nu|}{p+1} - \frac{1}{2}} \left(1 + \frac{u}{2x}\right) du = \int_0^\infty e^{-u} u^{\frac{|\nu|}{p+1} - \frac{1}{2}} du \in (0, \infty).$$

due to $x^{-\frac{1}{2}}$ being integrable on $[0, a]$ for any $a < \infty$. This shows that when $y \leq x$

$$\lim_{\lambda \rightarrow \infty} \lambda^{-\frac{1}{2}} \log Q_x^\delta [\exp(-\lambda \|X\|_{p,y}^p)] = -\frac{\sqrt{2}}{p+1} \left| x^{\frac{p+1}{2}} - y^{\frac{p+1}{2}} \right|.$$

In order to show the above limit when $y > x$, it suffices to show that $\lim_{x \rightarrow \infty} \frac{\log I_{\frac{\nu}{p+1}}(x)}{x} = 1$.

First suppose $\nu > 0$. Using the integral representation (2.6) we can obtain

$$I_\alpha(x) = \frac{\left(\frac{x}{2}\right)^\alpha}{\Gamma\left(\alpha + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_0^1 (1-t^2)^{\alpha-\frac{1}{2}} e^{-xt} dt + \frac{\left(\frac{2}{x}\right)^{\frac{1}{2}} e^x}{\Gamma\left(\alpha + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_0^x \left(1 - \frac{u}{2x}\right)^{\alpha-\frac{1}{2}} u^{\alpha-\frac{1}{2}} e^{-u} du,$$

where we used the change of variable $t = -1 + \frac{u}{x}$ to obtain the second integral above. Thus,

$$\begin{aligned} & \lim_{x \rightarrow \infty} \frac{\log I_{\frac{\nu}{p+1}}(x)}{x} = 1 \\ \iff & \lim_{x \rightarrow \infty} \frac{-\frac{1}{2} \log x + \log \left(e^{-x} \sqrt{x} I_{\frac{\nu}{p+1}}(x) \right)}{x} = 0 \\ \iff & \lim_{x \rightarrow \infty} \frac{\log \left(e^{-x} 2^{-\frac{\nu}{p+1}} x^{\frac{\nu}{p+1} + \frac{1}{2}} \int_0^1 (1-t^2)^{\frac{\nu}{p+1} - \frac{1}{2}} e^{-xt} dt + \sqrt{2} \int_0^x \left(1 - \frac{u}{2x}\right)^{\frac{\nu}{p+1} - \frac{1}{2}} u^{\frac{\nu}{p+1} - \frac{1}{2}} e^{-u} du \right)}{x} \\ & = 0, \end{aligned}$$

where the last equivalence follows from that $\lim_{x \rightarrow \infty} \frac{\log x}{x} = 0$. Moreover, the last equality holds since

$$\begin{aligned} & \lim_{x \rightarrow \infty} e^{-x} 2^{-\frac{\nu}{p+1}} x^{\frac{\nu}{p+1} + \frac{1}{2}} \int_0^1 (1-t^2)^{\frac{\nu}{p+1} - \frac{1}{2}} e^{-xt} dt + \sqrt{2} \int_0^x \left(1 - \frac{u}{2x}\right)^{\frac{\nu}{p+1} - \frac{1}{2}} u^{\frac{\nu}{p+1} - \frac{1}{2}} e^{-u} du \\ & = \sqrt{2} \int_0^\infty u^{\frac{\nu}{p+1} - \frac{1}{2}} e^{-u} du \in (0, \infty). \end{aligned}$$

Now, we turn to prove the limit for $\nu \leq 0$. Observe that since $K_\alpha(x)$ converges to 0 and $I_\alpha(x)$ diverges to ∞ as $x \rightarrow \infty$, the limits we have obtained so far imply for $\nu \geq 0$

$$\lim_{x \rightarrow \infty} \frac{K'_{\frac{\nu}{p+1}}(x)}{K_{\frac{\nu}{p+1}}(x)} = \lim_{x \rightarrow \infty} \frac{I'_{\frac{\nu}{p+1}}(x)}{I(x)} = 1,$$

i.e. for any $\varepsilon > 0$ there exists an x^* such that whenever $x > x^*$ we have

$$|K'_{\frac{\nu}{p+1}}(x) - K_{\frac{\nu}{p+1}}(x)| < \varepsilon K_{\frac{\nu}{p+1}}(x), \quad |I'_{\frac{\nu}{p+1}}(x) - I_{\frac{\nu}{p+1}}(x)| < \varepsilon I_{\frac{\nu}{p+1}}(x). \quad (3.1)$$

On the other hand, (2.5) yields

$$\lim_{x \rightarrow \infty} \frac{\log I_{-\frac{\nu}{p+1}}(x)}{x} = \lim_{x \rightarrow \infty} \frac{\log \left\{ I_{-\frac{\nu}{p+1}}(x) - \frac{2 \sin(\frac{\nu}{p+1}\pi)}{\pi} K_{\frac{\nu}{p+1}}(x) \right\}}{x}.$$

However, the above limit equals in view of L'Hospital's rule

$$\lim_{x \rightarrow \infty} \frac{I'_{-\frac{\nu}{p+1}}(x) - \frac{2 \sin(\frac{\nu}{p+1}\pi)}{\pi} K'_{\frac{\nu}{p+1}}(x)}{I_{-\frac{\nu}{p+1}}(x) - \frac{2 \sin(\frac{\nu}{p+1}\pi)}{\pi} K_{\frac{\nu}{p+1}}(x)},$$

which equals 1 since for any $x > x^*$

$$\left| \frac{I'_{-\frac{\nu}{p+1}}(x) - \frac{2 \sin(\frac{\nu}{p+1}\pi)}{\pi} K'_{\frac{\nu}{p+1}}(x)}{I_{-\frac{\nu}{p+1}}(x) - \frac{2 \sin(\frac{\nu}{p+1}\pi)}{\pi} K_{\frac{\nu}{p+1}}(x)} - 1 \right| < \varepsilon$$

by (3.1) and the fact that $K_\alpha \equiv K_{-\alpha}$ for all $\alpha \geq 0$. This completes the proof of the first assertion of the theorem.

The second assertion follows by applying de Bruijn's exponential Tauberian theorem (see Theorem 4.12.9 in [3]) to $\alpha = -1$ and $\beta = \frac{1}{2(p+1)^2} \left(x^{\frac{p+1}{2}} - y^{\frac{p+1}{2}} \right)^2$. \blacksquare

Observe that $\Sigma_{p,0,y}^\delta$ is an increasing process when indexed by y . We will next use the above theorem to obtain Chung's LIL for $(\Sigma_{p,0,y}^\delta)_{y \geq 0}$.

Theorem 3.2 *Let $\phi(y) := \frac{y^{p+1}}{\log \log y}$. Then, for any $\nu \geq -1$ and $p > 0$ one has*

$$\liminf_{y \rightarrow \infty} \frac{\Sigma_{p,0,y}^\delta}{\phi(y)} = \frac{1}{2(p+1)^2}, \quad Q_0^\delta\text{-a.s..}$$

PROOF. It follows from Theorem 3.1 that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \log Q_0^\delta [\Sigma_{p,0,1}^\delta < \varepsilon] = -\frac{1}{2(p+1)^2},$$

thus, for sufficiently small ε ,

$$Q_0^\delta [\Sigma_{p,0,1}^\delta < \varepsilon] \leq \exp \left(-\frac{K}{\varepsilon} \right),$$

where K is a fixed, but arbitrary, constant in $(0, \frac{1}{2(p+1)^2})$. Fix $C > 1$ and set $y_n = C^n$. Next choose $k > 0$ so that $kC^{p+1} < K$. In view of Propositions 2.1 and 2.2, we get for all large n

$$\begin{aligned} Q_0^\delta [\Sigma_{p,0,y_n}^\delta < k\phi(y_{n+1})] &= Q_0^\delta \left[\Sigma_{p,0,1}^\delta < \frac{kC^{p+1}}{\log \log y_{n+1}} \right] \\ &\leq \exp \left(-\frac{K}{kC^{p+1}} \log((n+1) \log C) \right) = (\log C)^{-\frac{K}{kC^{p+1}}} (n+1)^{-\frac{K}{kC^{p+1}}}, \end{aligned}$$

which is summable in n . Therefore, by the first Borel-Cantelli lemma, we have that, a.s. for large n , $\frac{\Sigma_{p,0,y_n}^\delta}{\phi(y_{n+1})} \geq k$. On the other hand, for $y \in [y_n, y_{n+1}]$,

$$\Sigma_{p,0,y}^\delta \geq \Sigma_{p,0,y_n}^\delta \geq k\phi(y_{n+1}) \geq k\phi(y),$$

which shows that

$$\liminf_{y \rightarrow \infty} \frac{\Sigma_{p,0,y}^\delta}{\phi(y)} \geq k, \quad Q_0^\delta\text{-a.s.},$$

and thus

$$\liminf_{y \rightarrow \infty} \frac{\Sigma_{p,0,y}^\delta}{\phi(y)} \geq \frac{1}{2(p+1)^2}, \quad Q_0^\delta\text{-a.s.},$$

by the arbitrariness of C, K and k .

We now turn to prove the reverse inequality. First, let's observe that

$$\liminf_{y \rightarrow \infty} \frac{\Sigma_{p,0,y}^\delta}{y^{p+1}} < \infty, \quad Q_0^\delta\text{-a.s.} \quad (3.2)$$

The above claim follows from a direct application of Fatou's lemma since $\Sigma_{p,0,y}^\delta \stackrel{d}{=} y^{p+1} \Sigma_{p,0,1}^\delta$.

Next, fix an $\varepsilon > 0$, let $y_n = n^n$ and consider the events

$$E_n := \left[\int_{R_{y_{n-1}}}^{R_{y_n}} X_s^p ds \leq (1 + 2\varepsilon) \frac{1}{2(p+1)^2} \phi(y_n) \right].$$

It follows from the strong Markov property of X that E_n s are independent, and we will now see that E_n s occur infinitely often due to the second Borel-Cantelli lemma. Indeed, by the definition of $\Sigma_{p,0,y_n}^\delta$, we obtain

$$\begin{aligned} Q_0^\delta(E_n) &\geq Q_0^\delta \left[\Sigma_{p,0,y_n}^\delta \leq (1 + 2\varepsilon) \frac{1}{2(p+1)^2} \phi(y_n) \right] \\ &= Q_0^\delta \left[\Sigma_{p,0,1}^\delta \leq (1 + 2\varepsilon) \frac{1}{2(p+1)^2 \log \log y_n} \right] \\ &\geq \exp \left(-\frac{1 + \varepsilon}{1 + 2\varepsilon} \log \log y_n \right) \geq \frac{1}{\log y_n}, \end{aligned}$$

where the second inequality is due to the fact that, for a given $\varepsilon > 0$, $Q_0^\delta [\Sigma_{p,0,1}^\delta \leq \eta] \geq \exp \left(-(1 + \varepsilon) \frac{1}{2(p+1)^2} \frac{1}{\eta} \right)$ for sufficiently small η in view of the convergence result of Theorem 3.1. Since $\frac{1}{n \log n}$ is not summable, it follows from the Borel-Cantelli lemma that E_n occurs infinitely often. As ε was arbitrary this allows us to conclude, a.s.,

$$\liminf_{n \rightarrow \infty} \frac{\int_{R_{y_{n-1}}}^{R_{y_n}} X_s^p ds}{\phi(y_n)} \leq \frac{1}{2(p+1)^2}.$$

Thus, Q_0^δ -a.s.,

$$\liminf_{n \rightarrow \infty} \frac{\Sigma_{p,0,y_n}^\delta}{\phi(y_n)} \leq \liminf_{n \rightarrow \infty} \frac{\Sigma_{p,0,y_{n-1}}^\delta}{\phi(y_n)} + \frac{1}{2(p+1)^2} = \liminf_{n \rightarrow \infty} \frac{\Sigma_{p,0,y_{n-1}}^\delta}{y_{n-1}^{p+1}} \frac{y_{n-1}^{p+1}}{\phi(y_n)} + \frac{1}{2(p+1)^2}. \quad (3.3)$$

On the other hand,

$$\frac{y_{n-1}^{p+1}}{\phi(y_n)} \leq \frac{\log \log n^{n(p+1)}}{n^{p+1}},$$

which converges to 0 as $n \rightarrow \infty$. Therefore, in view of (3.2) and (3.3), we obtain

$$\liminf_{n \rightarrow \infty} \frac{\Sigma_{p,0,y_n}^\delta}{\phi(y_n)} \leq \frac{1}{2(p+1)^2}, \quad Q_0^\delta\text{-a.s.}$$

■

4 Feller property and ‘time reversal’

In the previous section we have proved a law of iterated logarithm for $\Sigma_{p,0,y}^\delta$ by considering it as a process indexed by y . In this section we will see, for $\nu \geq 0$, that it is in fact an *inhomogeneous* Feller process and find its infinitesimal generator.

First of all, it immediately follows from the strong Markov property of X that $(\Sigma_{p,0,y}^\delta, \mathcal{F}_{R_y})_{y \geq 0}$ is Markov. Suppose $P_{x,y}$ is the associated semigroup, i.e. $P_{x,y}f(a) = Q_0^\delta[f(\Sigma_{p,0,y}^\delta) | \Sigma_{p,0,x}^\delta = a]$. Since the increments of $(\Sigma_{p,0,y}^\delta)_{y \geq 0}$ are independent, we have for any bounded measurable f

$$P_{x,y}f(a) = \int_0^\infty f(a+b) Q_x^\delta(\Sigma_{p,x,y}^\delta \in db). \quad (4.1)$$

Let C_0 denote the class of continuous functions on \mathbb{R}_+ that vanish at 0 and ∞ . (4.1) readily implies that when $f \in C_0$, $P_{x,y}f \in C_0$ as well. Moreover, it follows from Corollary 2.9, and the observation that u_1 is finite at 0, that for each $x \geq 0$ the measure $Q_x^\delta(\Sigma_{p,x,y}^\delta \in db)$ converges weakly to the Dirac point mass at 0 as $y \downarrow x$ since its Laplace transform converges to 1. Therefore, $\lim_{y \downarrow x} P_{x,y}f(a) = f(a)$ and consequently $(\Sigma_{p,0,y}^\delta, \mathcal{F}_{R_y})_{y \geq 0}$ is Feller.

The form of the infinitesimal generator of $(\Sigma_{p,0,y}^\delta, \mathcal{F}_{R_y})_{y \geq 0}$ will follow from the following theorem.

Theorem 4.1 *Suppose that $\nu \geq 0$. Then, for every $x \geq 0$ there exists a decreasing function $\bar{\pi}(x, \cdot)$ satisfying*

$$\int_0^\infty e^{-\frac{\lambda}{2}b} \bar{\pi}(x, b) db = \frac{2}{\lambda} \frac{w'(x)}{w(x)}, \quad \lambda \in (0, \infty),$$

where $w(x) := x^{-\frac{\nu}{2}} I_{\frac{\nu}{p+1}} \left(\frac{\sqrt{\lambda}}{p+1} x^{\frac{p+1}{2}} \right)$. Moreover,

$$\lim_{y \downarrow x} \frac{Q_0^\delta \left[\exp(-\lambda \Sigma_{p,0,y}^\delta) \mid \Sigma_{p,0,x}^\delta = a \right] - e^{-\lambda a}}{y - x} = \int_0^\infty \{e^{-\lambda(a+b)} - e^{-\lambda a}\} \pi(x, db), \quad (4.2)$$

where $\pi(x, db) := -\bar{\pi}(x, db)$ for $b \geq 0$. In particular,

$$\int_0^1 b\pi(x, db) < \infty.$$

PROOF. In view of the strong Markov property of X ,

$$Q_0^\delta \left[\exp \left(-\frac{\lambda}{2} \Sigma_{p,0,y}^\delta \right) \middle| \Sigma_{p,0,x}^\delta = a \right] = e^{-\frac{\lambda}{2}a} Q_x^\delta \left[\exp \left(-\frac{\lambda}{2} \Sigma_{p,x,y}^\delta \right) \right].$$

Thus, it follows from Corollary 2.9 that

$$\begin{aligned} \lim_{y \downarrow x} \frac{Q_0^\delta \left[\exp \left(-\frac{\lambda}{2} \Sigma_{p,0,y}^\delta \right) \middle| \Sigma_{p,0,x}^\delta = a \right] - e^{-\frac{\lambda}{2}a}}{y - x} &= e^{-\frac{\lambda}{2}a} \lim_{y \downarrow x} \frac{Q_x^\delta \left[\exp \left(-\frac{\lambda}{2} \Sigma_{p,x,y}^\delta \right) \right] - 1}{y - x} \\ &= e^{-\frac{\lambda}{2}a} \lim_{y \downarrow x} \frac{\frac{w(x)}{w(y)} - 1}{y - x} = -e^{-\frac{\lambda}{2}a} \frac{w'(x)}{w(x)}. \end{aligned} \quad (4.3)$$

On the other hand, using integration by parts, we obtain

$$\begin{aligned} Q_x^\delta \left[\exp \left(-\frac{\lambda}{2} \Sigma_{p,x,y}^\delta \right) \right] - 1 &= \int_0^\infty e^{-\frac{\lambda}{2}b} Q_x^\delta (\Sigma_{p,x,y}^\delta \in db) - 1 \\ &= -\frac{\lambda}{2} \int_0^\infty e^{-\frac{\lambda}{2}b} Q_x^\delta (\Sigma_{p,x,y}^\delta > b) db. \end{aligned} \quad (4.4)$$

It is well-known (see Corollary 3.8 in Chap. VII of [14]) that

$$Q_x^\delta [\Sigma_{p,x,y}^\delta] = \int_0^y z^p G(x, z) m(dz),$$

where m and G are the associated speed measure and Green's function, respectively. In our case, these are given by

$$m(dz) = \frac{z^\nu}{2\nu} dz; \quad G(x, z) = -y^{-\nu} + (x \vee z)^{-\nu}.$$

Consequently, the above formula yields

$$Q_x^\delta [\Sigma_{p,x,y}^\delta] = \frac{y^{p+1} - x^{p+1}}{2(p+1)(p+\nu+1)},$$

which in particular implies that $\Pi(x, y, db) := \frac{2(p+1)(p+\nu+1)}{y^{p+1} - x^{p+1}} Q_x^\delta (\Sigma_{p,x,y}^\delta > b) db$ is a probability measure on $[0, \infty)$ for each (x, y) . However, (4.3) and (4.4) imply that $\Pi(x, y, \cdot)$ converges weakly as y tends to x to some probability measure, $\bar{\Pi}(x, \cdot)$ on $[0, \infty)$ which satisfies¹

$$L \left(\frac{\lambda}{2} \right) := \int_0^\infty e^{-\frac{\lambda}{2}b} \bar{\Pi}(x, db) = \frac{4(p+\nu+1)}{\lambda} x^{-p} \frac{w'(x)}{w(x)}. \quad (4.5)$$

¹Using the integral representation of I_α for $\alpha > -\frac{1}{2}$, it is tedious but straightforward to check that this representation holds for $x = 0$ as well by taking the limit as $x \rightarrow 0$ and showing that $L(\lambda) < \infty$ for $\lambda > 0$.

Moreover, since $Q_x^\delta(\Sigma_{p,x,y}^\delta > b)$ is decreasing in b for each (x, y) , the limiting measure $\bar{\Pi}$ is necessarily of the form

$$c\varepsilon_0(db) + 2(p + \nu + 1)x^{-p}\bar{\pi}(x, b) db,$$

where ε_0 is the Dirac point mass at 0, c a nonnegative constant, and $\bar{\pi}(x, \cdot)$ is a decreasing function for each x . In particular, $\bar{\pi}(x, \infty) = 0$. In order to find the constant c , it suffices to check the value of the function L at ∞ . However, using the explicit form of w ,

$$c = \lim_{\lambda \rightarrow \infty} \frac{4(p + \nu + 1)}{\lambda} x^{-p} \frac{w'(x)}{w(x)} = \lim_{\lambda \rightarrow \infty} 2 \frac{p + \nu + 1}{\sqrt{\lambda} x^{\frac{p+1}{2}}} \frac{I'_{\frac{\nu}{p+1}}\left(\frac{\sqrt{\lambda}}{p+1} x^{\frac{p+1}{2}}\right)}{I_{\frac{\nu}{p+1}}\left(\frac{\sqrt{\lambda}}{p+1} x^{\frac{p+1}{2}}\right)} = 0$$

since for any $\alpha > -\frac{1}{2}$

$$\frac{1}{x} \frac{I'_\alpha(x)}{I_\alpha(x)} = \alpha \frac{1}{x^2} - \frac{1}{x} \frac{\int_{-1}^1 (1-t^2)^{\alpha-\frac{1}{2}} t e^{-xt} dt}{\int_{-1}^1 (1-t^2)^{\alpha-\frac{1}{2}} e^{-xt} dt}$$

and that

$$\left| \frac{\int_{-1}^1 (1-t^2)^{\alpha-\frac{1}{2}} t e^{-xt} dt}{\int_{-1}^1 (1-t^2)^{\alpha-\frac{1}{2}} e^{-xt} dt} \right| \leq 1.$$

Thus, we have shown that

$$-\frac{w'(x)}{w(x)} = \lim_{y \downarrow x} \frac{Q_x^\delta [\exp(-\frac{\lambda}{2} \Sigma_{p,x,y}^\delta)] - 1}{y - x} = -\frac{\lambda}{2} \int_0^\infty e^{-\frac{\lambda}{2} b} \bar{\pi}(x, b) db.$$

Since $\bar{\pi}$ is decreasing with $\bar{\pi}(x, \infty) = 0$, we obtain by integrating by parts

$$\lim_{y \downarrow x} \frac{Q_0^\delta \left[\exp\left(-\frac{\lambda}{2} \Sigma_{p,0,y}^\delta\right) \middle| \Sigma_{p,0,x}^\delta = a \right] - e^{-\frac{\lambda}{2} a}}{y - x} = e^{-\frac{\lambda}{2} a} \int_0^\infty \left\{ e^{-\frac{\lambda}{2} b} - 1 \right\} \pi(x, db)$$

where $\pi(x, db) = -\bar{\pi}(x, db)$. Finally, note that one necessarily has

$$\int_0^1 b \pi(x, db) < \infty,$$

since otherwise L , as defined in (4.5), would have been infinite. ■

The above theorem yields that the sequence of measures $\left(\frac{Q_x^\delta(\Sigma_{p,x,y}^\delta > b)}{y-x} db \right)$ converges vaguely to a finite measure on $(0, \infty)$ as $y \rightarrow x$. Thus, for any $f \in C_K^1(\mathbb{R}_+, \mathbb{R})$, i.e. the space of continuously differentiable functions with a compact support, we have

$$\lim_{y \downarrow x} \frac{Q_0^\delta [f(\Sigma_{p,0,y}^\delta) | \Sigma_{p,0,x}^\delta = a] - f(a)}{y - x} = \int_0^\infty \{f(a+b) - f(a)\} \pi(x, db). \quad (4.6)$$

In other words, letting $B(\mathbb{R}_+)$ denote the bounded Borel functions defined on \mathbb{R}_+ , if we define the operator $\mathcal{A}_x : B(\mathbb{R}_+) \mapsto B(\mathbb{R}_+)$ by setting

$$\mathcal{A}_x f(a) := \int_0^\infty \{f(a+b) - f(a)\} \pi(x, db) \quad \text{for } f \in C_K^1(\mathbb{R}_+, \mathbb{R}),$$

then we see that the process $(M_y^f)_{y \geq 0}$ defined by

$$M_y^f := f(\Sigma_{p,0,y}^\delta) - \int_0^y \mathcal{A}_x f(\Sigma_{p,0,x}^\delta) dx$$

is a martingale with respect to the filtration $(\mathcal{F}_{R_y})_{y \geq 0}$ whenever f belongs to the domain of \mathcal{A}_x for all $x \geq 0$. The form of the infinitesimal generator also reveals the fact that the increasing process $(\Sigma_{p,0,y}^\delta)_{y \geq 0}$ is purely discontinuous, i.e. there is no interval (a, b) in which it is continuous.

Remark 5 *It follows from the fact that $(R_y)_{y \geq 0}$ is left continuous that $(\Sigma_{p,0,y}^\delta)_{y \geq 0}$ is a left-continuous process. However, in view of the above Feller property one can obtain a càdlàg version of it when we augment the filtration with the null sets. Existence of a right-continuous version can also be independently verified by observing that*

$$\lim_{y \downarrow x} Q_0^\delta [\exp(-\lambda \Sigma_{p,0,y}^\delta)] = Q_0^\delta [\exp(-\lambda \Sigma_{p,0,x}^\delta)].$$

We will end this section by analysing a specific ‘time reversal’ example. To this end let $L_x := \sup\{t \geq 0 : X_t = x\}$ and suppose that $\nu > 0$ so that $Q_0^\delta(L_x < \infty) = 1$ for all $x \geq 0$. We will consider the process Z^δ defined by

$$Z_x^\delta := \int_{L_1-x}^{L_1} X_s^p ds \quad \forall x \in [0, 1). \quad (4.7)$$

In view of the well-known time reversal results for diffusions, see, e.g., Exercise 1.23 in Chap. XI of [14], the law of the process $(X_{L_1-t}, t < L_1)$ under Q_0^δ is identical to that of $(X_t, t < R_0)$ under $Q_1^{2-2\nu}$. Recall that $Q_1^{2-2\nu}(R_0 < \infty) = 1$. Thus, we can write

$$Z_x^\delta = \int_0^{R_1-x} X_s^p ds, \quad X = BESQ^{2-2\nu}(1). \quad (4.8)$$

Note that the above equality is to be understood in the sense of equality between the laws of the processes. Due to the strong Markov property of X we again have that $(Z_x^\delta)_{x \in [0,1]}$ is a Markov process with respect to the filtration $(\mathcal{F}_x)_{x \in [0,1]}$ where $\mathcal{F}_x := \sigma(X_s; L_1-x \leq s \leq L_1)$. Observe, more easily in view of (4.8), that Z^δ , too, has independent increments rendering its Feller property in view of the arguments that led to the Feller property of $(\Sigma_{p,0,x}^\delta)_{x \geq 0}$ at the beginning of this section. The next theorem will yield the form of the infinitesimal generator. Its proof will follow similar lines of the proof of Theorem 4.1 so we will only give the details when it differs.

Theorem 4.2 *Let Z^δ be as defined in (4.7) and suppose $\nu \in (0, 1]$. Then, for every $x \in [0, 1)$ there exists a decreasing function $\tilde{\pi}(x, \cdot)$ satisfying*

$$\int_0^\infty e^{-\frac{\lambda}{2}b} \tilde{\pi}(x, b) db = -\frac{2}{\lambda} \frac{w'(1-x)}{w(1-x)}, \quad \lambda \in (0, \infty),$$

where $w(x) := x^{\frac{\nu}{2}} K_{\frac{\nu}{p+1}} \left(\frac{\sqrt{\lambda}}{p+1} x^{\frac{p+1}{2}} \right)$. Moreover,

$$\lim_{y \downarrow x} \frac{Q_0^\delta [\exp(-\lambda Z_y^\delta) | Z_x^\delta = a] - e^{-\lambda a}}{y - x} = \int_0^\infty \{e^{-\lambda(a+b)} - e^{-\lambda a}\} \rho(x, db), \quad (4.9)$$

where $\rho(x, db) := -\tilde{\pi}(x, db)$ for $b \geq 0$. In particular,

$$\int_0^1 b \rho(x, db) < \infty.$$

PROOF. First observe that in view of (4.8) and the aforementioned independent increments property

$$Q_0^\delta \left[\exp \left(-\frac{\lambda}{2} Z_y^\delta \right) \middle| Z_x^\delta = a \right] = e^{-\frac{\lambda}{2}a} Q_{1-x}^\beta \left[\exp \left(-\frac{\lambda}{2} \Sigma_{p,1-x,1-y}^\beta \right) \right],$$

where $\beta = 2-2\nu$. Therefore, the same arguments in the beginning of the proof of Theorem 4.1 yields that the measures $\frac{Q_{1-x}^\beta(\Sigma_{p,1-x,1-y}^\beta > b) db}{y-x}$ converges vaguely to some probability measure, $\tilde{\Pi}(x, \cdot)$ on $[0, \infty)$ which satisfies

$$L \left(\frac{\lambda}{2} \right) := \int_0^\infty e^{-\frac{\lambda}{2}b} \tilde{\Pi}(x, db) = -\frac{1}{\lambda} \frac{w'(1-x)}{w(1-x)}. \quad (4.10)$$

Moreover, the limiting measure is necessarily of the form

$$c\varepsilon_0(db) + \tilde{\pi}(x, b) db,$$

where ε_0 is the Dirac point mass at 0, c a nonnegative constant, and $\tilde{\pi}(x, \cdot)$ is a decreasing function for each x . Observe that since $Q_{1-x}^\beta(\Sigma_{p,1-x,1-y}^\beta) = \infty$, the measures $\frac{Q_{1-x}^\beta(\Sigma_{p,1-x,1-y}^\beta > b) db}{y-x}$ are not finite and neither is their vague limit. On the other hand, we can still conclude that $c = 0$ since, in view of (2.10), for any $\alpha > -\frac{1}{2}$ one has

$$\frac{1}{x} \frac{K'_\alpha(x)}{K_\alpha(x)} = -\frac{1}{2} \frac{1}{x^2} - \frac{1}{x} - \frac{\alpha - \frac{1}{2}}{x^3} \frac{\int_0^\infty e^{-u} u^{\alpha+\frac{1}{2}} \left(1 + \frac{u}{2x}\right)^{\alpha-\frac{3}{2}} du}{\int_0^\infty e^{-u} u^{\alpha-\frac{1}{2}} \left(1 + \frac{u}{2x}\right)^{\alpha-\frac{1}{2}} du},$$

which converges to 0 as $x \rightarrow \infty$. Next, integrating (4.10) by parts yields

$$\tilde{\pi}(x, \infty) - \int_0^\infty (1 - e^{-\frac{\lambda}{2}b}) \tilde{\pi}(x, db) = -\frac{w'(1-x)}{w(1-x)}. \quad (4.11)$$

Thus, due to the dominated convergence theorem, taking the limit of the above as λ tends to 0 yields

$$\tilde{\pi}(x, \infty) = -\lim_{\lambda \rightarrow 0} \frac{w'(1-x)}{w(1-x)}.$$

However, using the representation in (2.9) one has that

$$x \frac{K'_\alpha(x)}{K_\alpha(x)} = -\alpha - 2 \left(\alpha - \frac{1}{2} \right) x^2 \frac{\int_x^\infty e^{-u} (u^2 - x^2)^{\alpha - \frac{3}{2}} du}{\int_x^\infty e^{-u} (u^2 - x^2)^{\alpha - \frac{1}{2}} du},$$

for any $\alpha > -\frac{1}{2}$. However, $x^2 \int_x^\infty e^{-u} (u^2 - x^2)^{\alpha - \frac{3}{2}} du$ converges to 0 as x tends to 0 due to monotone convergence theorem if $\alpha > 3/2$ or dominated convergence theorem otherwise. This in turn yields that

$$\lim_{\lambda \rightarrow 0} \frac{w'(x)}{w(x)} = \frac{\nu}{2x} + \frac{p+1}{2x} \lim_{\lambda \rightarrow 0} \frac{K'_{\frac{\nu}{p+1}} \left(\frac{\sqrt{\lambda}}{p+1} x^{\frac{p+1}{2}} \right)}{K_{\frac{\nu}{p+1}} \left(\frac{\sqrt{\lambda}}{p+1} x^{\frac{p+1}{2}} \right)} \frac{\sqrt{\lambda}}{p+1} x^{\frac{p+1}{2}} = 0,$$

and thus $\tilde{\pi}(x, \infty) = 0$. As in the proof of Theorem 4.1

$$e^{-\frac{\lambda}{2}a} \frac{w'(1-x)}{w(1-x)} = \lim_{y \downarrow x} \frac{Q_0^\delta [\exp(-\frac{\lambda}{2}Z_y^\delta) | Z_x^\delta = a] - e^{-\frac{\lambda}{2}a}}{y - x}.$$

Thus, combining above with (4.11) and plugging in the value of $\tilde{\pi}(x, \infty)$ yield

$$\lim_{y \downarrow x} \frac{Q_0^\delta [\exp(-\frac{\lambda}{2}Z_y^\delta) | Z_x^\delta = a] - e^{-\lambda a}}{y - x} = \int_0^\infty \left\{ e^{-\frac{\lambda}{2}(a+b)} - e^{-\frac{\lambda}{2}a} \right\} \rho(x, db).$$

■

Example 4.1 As an application of the above theorem consider the case when $\nu \in (0, 1]$ and $\frac{\nu}{p+1} = \frac{1}{2}$. Then, the associated $\tilde{\pi}$ is defined by

$$\int_0^\infty e^{-\frac{\lambda}{2}b} \tilde{\pi}(x, b) db = \frac{1}{\sqrt{\lambda}} (1-x)^{\nu-1}$$

in view of the explicit form for $K_{\frac{1}{2}}$. Thus, by inverting the above transform, we have

$$\tilde{\pi}(x, b) = (1-x)^{\nu-1} \frac{1}{\sqrt{2\pi b}}.$$

This reveals that the infinitesimal generator, \tilde{A}_x of Z^δ is defined by

$$\tilde{A}_x = (1-x)^{\nu-1} \int_0^\infty \{f(a+b) - f(a)\} \frac{1}{2\sqrt{2\pi b^3}} db$$

for any f in C^1 with a compact support. Consequently,

$$f(Z_x^\delta) - \int_0^x (1-y)^{\nu-1} \int_0^\infty \{f(Z_y^\delta + b) - f(Z_y^\delta)\} \frac{1}{2\sqrt{2\pi}b^3} db dy$$

is an $\overleftarrow{\mathcal{F}}$ -martingale for such f .

Example 4.2 Observe that although Z^δ , or $(\Sigma_{p,0,y}^\delta)$, is an increasing process with independent increments, it is not a subordinator (see [2] for a definition and further properties) since the increments are not stationary. However, in the above example, if one takes $\nu = 1$, then one sees that Z^δ becomes time homogeneous, i.e. Z^δ is a subordinator. Observe that $\nu = 1$ implies $p = 1$ in this framework. More precisely,

$$Z_x^\delta = Z_x^4 = \int_{L_1-x}^{L_1} X_s ds,$$

where X is $BESQ^4(1)$, is a subordinator. Moreover, Corollary 2.4 and (4.8) yield

$$Q_0^\delta (\exp (-\lambda Z_x^4)) = \exp \left(-\sqrt{\frac{\lambda}{2}} x \right).$$

Thus, $Z_x^4 \stackrel{d}{=} T_{\frac{x}{2}}$, where T_x is the first hitting time of x for a Brownian motion starting at 0.

5 Applications to finance

Our aim in this section is to give some examples arising from some financial models and discuss how the results from previous sections can be used to obtain prices of certain financial products.

As explained in Introduction the process X is commonly used in the finance literature to model interest rates. Suppose the spot interest rate is given by X^p where $p > -1$ and X is $BESQ^\delta(x)$ and consider the following *exotic* derivative on interest rates which pays one unit of a currency at time R_y if the accumulated interest is less than k , i.e. $\Sigma_{p,x,y}^\delta \leq \log k$. This is an example of a digital option and its price, as usual in the Finance Theory, is given as an expectation of its discounted payoff:

$$D(k; \delta, p, x, y) = Q_x^\delta \left[\mathbf{1}_{[\Sigma_{p,x,y}^\delta \leq k]} \exp (-\Sigma_{p,x,y}^\delta) \right].$$

On the other hand, if one computes the Laplace transform of $D(k; \delta, p, x, y)$, one obtains

$$\int_0^\infty e^{-\mu k} D(k; \delta, p, x, y) dk = Q_x^\delta \left[\int_{\Sigma_{p,x,y}^\delta}^\infty e^{-\mu k} \exp (-\Sigma_{p,x,y}^\delta) dk \right] = Q_x^\delta \left[\exp (-(\mu + 1)\Sigma_{p,x,y}^\delta) \right],$$

which is at our disposal due to the results from Section 2. In particular, the above identity implies

$$\int_0^z e^u D(u; \delta, p, x, y) du = Q_x^\delta [\Sigma_{p,x,y}^\delta \leq z], \quad \forall z \geq 0. \quad (5.1)$$

Moreover, once the function D is determined by inverting the corresponding Laplace transform, one can also obtain the prices for put options written on the accumulated interest. Indeed, for any $K > 1$

$$\begin{aligned} P(\delta; x, y, K) &:= Q_x^\delta \left[(K - \exp(\Sigma_{p,x,y}^\delta))^+ \exp(-\Sigma_{p,x,y}^\delta) \right] \\ &= Q_x^\delta \left[\int_1^K \mathbf{1}_{[\Sigma_{p,x,y}^\delta \leq \log k]} dk \exp(-\Sigma_{p,x,y}^\delta) \right] \\ &= \int_1^K D(\log k; \delta, p, x, y) dk = \int_0^{\log K} e^u D(u; \delta, p, x, y) du. \end{aligned} \quad (5.2)$$

Note that for $K = 1$, i.e. when the option is *at-the-money* since the cumulative interest at $t = 0$ is defined to be 1, the put option is worthless. We can in fact get how fast the option becomes worthless as K approaches to 1. Indeed, comparing (5.2) to (5.1) yields the following asymptotics for the option value in view of Theorem 3.1:

$$\lim_{K \downarrow 1} \log K \log P(\delta; x, y, K) = -\frac{1}{2(p+1)^2} \left(x^{\frac{p+1}{2}} - y^{\frac{p+1}{2}} \right)^2. \quad (5.3)$$

The above expression tells us how small the option price becomes when the option is slightly *in-the-money*, i.e. when K is very close to 1.

Next, assume $y < x$ and consider another type of a put option on the maximum of the short rate with maturity R_y and payoff $(K - \max_{t \leq R_y} X_t)^+$ for $K > x$. The price of this option equals

$$\begin{aligned} Q_x^\delta \left[\exp(-\Sigma_{p,x,y}^\delta) \int_0^K \mathbf{1}_{[\max_{t \leq R_y} X_t < a]} da \right] &= Q_x^\delta \left[\exp(-\Sigma_{p,x,y}^\delta) \int_x^K \mathbf{1}_{[R_a > R_y]} da \right] \\ &= \frac{u_0(\sqrt{x})}{u_0(\sqrt{y})} \left(\frac{x}{y} \right)^{-\frac{\nu}{2}} \int_x^K \frac{\tilde{s}_0(x) - \tilde{s}(a)}{\tilde{s}_0(y) - \tilde{s}_0(a)} da, \end{aligned}$$

in view of Corollary 2.3 and Proposition 2.3, where the pair (u_0, \tilde{s}_0) is computed by setting $\lambda = 2$.

Finally, if one is interested in pricing an Asian option on the short rate until time R_y , it suffices to use the identity

$$Q_x^\delta \left[\left(K - \frac{\Sigma_{p,x,y}^\delta}{R_y} \right)^+ \exp(-\Sigma_{p,x,y}^\delta) \right] = Q_x^\delta \left[\exp(-\Sigma_{p,x,y}^\delta) \int_0^K \mathbf{1}_{[\Sigma_{p,x,y}^\delta < k R_y]} dk \right]$$

and invert the joint Laplace transform of R_y and $\Sigma_{p,x,y}^\delta$ obtained in Section 2.

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